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THE SOLUTION OF SOME FRACTIONAL DIFFERENTIAL EQUATIONS BY SUMUDU TRANSFORM METHOD

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ABSTRACT
In this paper, the solutions of some fractional differential equations are obtained by using Sumudu transform. The fractional derivative is considered in the Caputo sense. Some illustrative examples are also given.

Keywords: *Fractional derivatives; Sumudu transform; Fractional differential equations.*

1 INTRODUCTION
The fractional differential equations occurs in applied mathematics, physics, engineering sciences and other branches of knowledge. Fractional integrals and derivatives have various definitions and take various names in the literature. In this paper, we apply the Sumudu transform to some fractional differential equations of Caputo type and give some illustrative examples.

The Sumudu transform is an integral transform introduced by Watugala G. K. in the year 1990 to solve differential equations and control engineering problems [4]. The Sumudu transform can be used to solve many types of difference and differential equations without restoring to a new frequency domain.

2 PRELIMINARY RESULTS, NOTATIONS AND TERMINOLOGY
In this section we give definitions and some basic results which are used in the paper.

Definition 2.1 : Consider a set A defined as

$$A = \{ f(t) \mid \exists M, \tau_1, \tau_2 > 0, | f(t) | \leq M e^{\frac{|t|}{\tau_j}} \text{ if } t \in (-1)^j \times [0,\infty) \}. \quad (1)$$

For all real $t \geq 0$, the Sumudu transform of a function $f(t) \in A,$ denoted by $F(u) = S[f(t)]$, is defined as

$$F(u) = S[f(t)](u) = \int_0^\infty \frac{1}{u} e^{\frac{-t}{u}} f(t) dt, u \in (-\tau_1, \tau_2) \quad (2)$$

The function $f(t)$ in equation (2) is called the inverse Sumudu transform of $F(u)$ and is denoted by $f(t) = S^{-1}[F(u)].$

Definition 2.2: A real function $f(t)$, $t \geq 0$ is said to be in space C_μ, $\mu \in \mathbb{R}$ if there exists a real number $n (> \mu)$, such that $f(t) = t^n f_1(t)$, where $f_1(t) \in C[0,\infty)$, and is said to be in space C_μ^k if and only if $f^{(k)} \in C_\mu$, $k \in \mathbb{N}$.[3]

Definition 2.3 : The Caputo's fractional derivative of f is given by [1]

$$ {}_a^c D_t^\alpha f(t) = \frac{1}{\Gamma(\alpha - n)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \quad (3)$$

where $f \in C_{-1}^n, n-1 < \alpha \leq n, n \in \mathbb{N}.$

Theorem 2.1: If for a positive integer n, $-1 < n-1 < \alpha \leq n$, and $F(u)$ is the Sumudu transform of the function $f(t)$, then the Sumudu transform of the Caputo fractional derivative of order α, denoted as $S[{}_a^c D_t^\alpha f(t)] = F_\alpha^c(u)$, is given by [7]

$$F_\alpha^c(u) = S[{}_a^c D_t^\alpha f(t)] = u^{-\alpha} [F(u) - \sum_{k=0}^{n-1} u^k [f^{(k)}(t)]_{t=0}], -1 < n-1 < \alpha \leq n, \quad (4)$$

where n is the smallest integer greater than α.

Theorem 2.2: Let $F(u)$ and $G(u)$ be the Sumudu transforms of $f(t)$ and $g(t)$ respectively. If

$$h(t) = (f(t) * g(t)) = \int_0^t f(\tau)g(t-\tau)d\tau$$

where * denotes convolutionof f and g, then the Sumudu transform of $h(t)$ is [5]

$$S[h(t)] = uF(u)G(u).$$

3 Application of Sumudu Transform To The Fractional Differential Equations [6]. **:**
Now, we obtain the solution of some fractional differential equations using Sumudu transform method .

Example (1) We consider fractional differential equation of the form [2]

$$D^{\alpha} y(t) = f(t) \tag{3.1}$$

with initial conditions $y^{(k)}(0) = c_k, k = 0,1,2,3,\ldots n-1$, where n is the smallest integer greater than α such that $-1 < n-1 < \alpha \le n$.

Solution: Let $f(t) \in A$. Applying the Sumudu transform on both side of equation (5) we get,

$$u^{-\alpha}Y(u) - \sum_{k=0}^{n-1} u^{k-\alpha} y^{(k)}(0) = F(u). \tag{3.2}$$

Using initial conditions and simplying,we have

$$Y(u) = u^{\alpha}\{F(u) + \sum_{k=0}^{n-1} c_k u^{k-\alpha}\}. \tag{3.3}$$

Taking the inverse Sumudu transform and using convolution theorem, we get

$$y(t) = \sum_{k=0}^{n-1} c_k \frac{1}{\Gamma(k+1)} t^{(k)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \tag{3.4}$$

which is the solution of equation (3.1).

Case (I) : Consider the Fractional Differential equation

$$D^{\frac{3}{2}} y(t) = 1 \tag{3.5}$$

with initial conditions $y(0) = 0$, $y'(0) = 0$.

Solution: Applying the Sumudu transform on both side of equation (3.5), we get,

$$S(D^{\frac{3}{2}} y(t)) = S(1)$$

$$u^{\frac{-3}{2}} Y(u) - u^{\frac{-3}{2}} y(0) - u^{\frac{-1}{2}} y'(0) = 1$$

$$u^{\frac{-3}{2}} Y(u) = 1$$

$$Y(u) = u^{\frac{3}{2}}.$$

Applying the inverse Sumudu transform ,we get

$$y(t) = \frac{t^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}$$

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$$y(t) = \frac{4}{3} t \sqrt{\frac{t}{\pi}}$$

which is the solution of equation (3.5).

Case (II) : Consider the Fractional Differential equation

$$D^{\frac{1}{2}} y(t) = e^t, \qquad (3.6)$$

with initial condition y(0)=1.

Solution : Applying the Sumudu transform on both side of equation (3.6), we get

$$S(D^{\frac{1}{2}} y(t)) = S(e^t)$$

$$u^{\frac{-1}{2}} Y(u) - u^{\frac{-1}{2}} y(0) = \frac{1}{1-u}$$

$$u^{\frac{-1}{2}} Y(u) - u^{\frac{-1}{2}} = \frac{1}{1-u}$$

$$Y(u) = 1 + \frac{\sqrt{u}}{1-u}.$$

Applying the inverse Sumudu transform, we get

$$y(t) = 1 + e^t erf(\sqrt{t}),$$

which is the required solution of equation (3.6).

Case (III) : Consider the Fractional Differential equation

$$D^{\frac{1}{2}} y(t) = \sin t, \qquad (3.7)$$

with initial condition $y(0) = 1$.

Solution : Applying the Sumudu transform on both side of equation (3.7), we get,

$$S(D^{\frac{1}{2}} y(t)) = S(\sin(t))$$

$$u^{\frac{-1}{2}} Y(u) - u^{\frac{-1}{2}} y(0) = \frac{u}{1+u^2}$$

$$u^{\frac{-1}{2}} Y(u) - u^{\frac{-1}{2}} = \frac{u}{1+u^2}$$

$$y(u) = 1 + \frac{u^{\frac{3}{2}}}{1+u^2}.$$

Applying the inverse Sumudu transform and using convolution theorem, we get

$$y(t) = S^{-1}(1) + S^{-1}[(u)(\sqrt{u})(\frac{1}{1+u^2})]$$

$$y(t) = 1 + \int_0^t 2\sqrt{\frac{\tau}{\pi}} \cos(t-\tau) d\tau.$$

which is the required solution of equation (3.7).

Example (2) Consider fractional differential equation of the form [2]

$$D^{\alpha} y(t) + D^{\beta} y(t) = f(t), \qquad (3.8)$$

with initial conditions $y^{(k)}(0) = c_k, k = 0, 1, 2, \ldots, l-1$, where α and β are positive numbers with $\alpha > 0$, $\beta > 0$, $l-1 < \alpha \le l$, $l \in N$, $0 < \beta < \alpha$ and $\alpha - l + 1 \ge \beta$ and l is the smallest integer greater than α.

Solution : Let $f(t) \in A$. Applying the Sumudu transform on both side of equation (3.9), we get

$$(u^{-\alpha} + u^{-\beta})Y(u) - \sum_{r=0}^{l-1} u^{r-\alpha} y^{(r)}(0) - \sum_{k=0}^{m-1} u^{k-\beta} y^{(k)}(0) = F(u), \qquad (3.9)$$

here m is the smallest integer greater than β, thus

$$Y(u) = (u^{-\alpha} + u^{-\beta})^{-1}\{F(u) + \sum_{r=0}^{l-1} u^{r-\alpha} y^{(r)}(0) + \sum_{k=0}^{m-1} u^{k-\beta} y^{(k)}(0)\} \qquad (3.10)$$

Applying the inverse Sumudu transform to the equation (3.8) and using initial conditions, we get

$$y(t) = S^{-1}\{(u^{-\alpha} + u^{-\beta})^{-1}[F(u) + \sum_{r=0}^{l-1} c_r u^{r-\alpha} + \sum_{k=0}^{m-1} c_k u^{k-\beta}]du\} \qquad (3.11)$$

which is the solution of equation (3.8).

Case (I) : Consider the Fractional Differential equation

$$D^{\frac{3}{2}} y(t) + D^{\frac{1}{2}} y(t) = t, \qquad (3.12)$$

with initial conditions $y(0) = 0$, $y'(0) = 0$.

Solution : Applying the Sumudu transform on both side of equation (3.12), we get

$$S[D^{\frac{3}{2}} y(t)] + S[D^{\frac{1}{2}} y(t)] = S(t)$$

$$u^{\frac{-3}{2}} Y(u) - u^{\frac{-3}{2}} y(0) - u^{\frac{-1}{2}} y'(0) + u^{\frac{-1}{2}} Y(u) - u^{\frac{-1}{2}} y(0) = u$$

$$(u^{\frac{-3}{2}} + u^{\frac{-1}{2}})Y(u) = u$$

$$Y(u) = \frac{u^3}{u^{\frac{3}{2}} + u^{\frac{1}{2}}}$$

$$Y(u) = u^2 \frac{\sqrt{u}}{u+1}.$$

Applying the inverse Sumudu transform and using the convolution theorem, we get

$$y(t) = S^{-1}[(u)(u)(\frac{\sqrt{u}}{1+u})]$$

$$y(t) = (-i)\int_0^t e^{-\tau} erf(i\sqrt{\tau})(t-\tau)d\tau$$

which is the solution of equation (3.12).

Case (II) : Consider the Fractional Differential equation

$$D^{\frac{3}{2}} y(t) + D^{\frac{1}{2}} y(t) = \sin t, \qquad (3.13)$$

with initial conditions $y(0)=0$, $y'(0)=0$.

Solution: Applying the Sumudu transform on both side of equation (3.13), we get

$$S[D^{\frac{3}{2}}y(t)]+S[D^{\frac{1}{2}}y(t)]=S(\sin t)$$

$$u^{\frac{-3}{2}}Y(u)-u^{\frac{-3}{2}}y(0)-u^{\frac{-1}{2}}y'(0)+u^{\frac{-1}{2}}Y(u)-u^{\frac{-1}{2}}y(0)=\frac{u}{1+u^2}$$

$$(u^{\frac{-3}{2}}+u^{\frac{-1}{2}})Y(u)=\frac{u}{1+u^2}$$

$$Y(u)=\frac{u^{\frac{3}{2}}}{(1+u)}\frac{u}{(1+u^2)}$$

$$Y(u)=u\frac{\sqrt{u}}{(u+1)}\frac{u}{(1+u^2)}.$$

Applying the inverse Sumudu transform and the convolution theorem, we get

$$y(t)=S^{-1}[(u)(\frac{\sqrt{u}}{1+u})(\frac{u}{1+u^2})]$$

$$y(t)=(-i)\int_0^t e^{-\tau}erf(i\sqrt{\tau})\sin(t-\tau)d\tau$$

which is the solution of equation (3.13).

Case (III) : Consider the Fractional Differential equation

$$D^{\frac{3}{2}}y(t)+D^{\frac{1}{2}}y(t)=e^t, \qquad (3.14)$$

with initial conditions $y(0)=1$, $y'(0)=2$.

Solution : Applying the Sumudu transform on both side of equation (3.14), we get

$$S[D^{\frac{3}{2}}y(t)]+S[D^{\frac{1}{2}}y(t)]=S(e^t)$$

$$u^{\frac{-3}{2}}Y(u)-u^{\frac{-3}{2}}y(0)-u^{\frac{-1}{2}}y'(0)+u^{\frac{-1}{2}}Y(u)-u^{\frac{-1}{2}}y(0)=\frac{1}{1-u}$$

$$(u^{\frac{-3}{2}}+u^{\frac{-1}{2}})Y(u)-u^{\frac{-3}{2}}-2u^{\frac{-1}{2}}-u^{\frac{-1}{2}}=\frac{1}{1-u}$$

$$Y(u)=(\frac{1}{1+u}+\frac{3u}{1+u}+\frac{u^{\frac{3}{2}}}{1-u^2})$$

Applying the inverse Sumudu transform and using the convolution theorem, we get

$$y(t)=S^{-1}(\frac{1}{1+u}+\frac{3u}{1+u}+\frac{u^{\frac{3}{2}}}{1-u^2})$$

$$y(t)=S^{-1}(\frac{1}{1+u})+S^{-1}(u\cdot 3\cdot\frac{1}{1+u})+S^{-1}(u\cdot\sqrt{u}\cdot\frac{1}{1-u^2})$$

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$$y(t) = e^{-t} + \int_0^t 3e^{-(t-\tau)} d\tau + \int_0^t 2\sqrt{\frac{\tau}{\pi}} \cdot \cosh(t-\tau) d\tau$$

which is the solution of equation (3.14).

4 CONCLUSION

The Sumudu transform is an useful operational transform method which plays an important role in treating homogenous and non-homogeneous fractional differential equations of Caputo type.

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